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Generalized Arcsine Law and Stable Law in an Infinite Measure Dynamical System

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Abstract Limit theorems for the time average of some observation functions in an infinite measure dynamical system are studied. It is known that intermittent phenomena, such as the Rayleigh-Benard convection and Belousov-Zhabotinsky reaction, are described by infinite measure dynamical systems. We show that the time average of the observation function which is not the $L^1(m)$ function, whose average with respect to the invariant measure m is finite, converges to the generalized arcsine distribution. This result leads to the novel view that the correlation function is intrinsically random and does not decay. Moreover, it is also numerically shown that the time average of the observation function converges to the stable distribution when the observation function has the infinite mean.

Keywords Non-stationary chaos · Infinite measure · Generalized arcsine law · Stable law · Non-equilibrium state

1 Introduction

Recently, the $1/f$ power spectrum and the power law phenomena, which are closely related to the intermittent phenomena [1], have been studied in various systems. Examples are the $1/f$ power spectrum in the Rayleigh-Benard convection [2], Belousov-Zhabotinsky reaction [3], fluorescence intermittency in single nanocrystals [4] and the power law decay of the earthquake phenomena [5, 6]. It is also known that such power law phenomena are clearly observed in Hamiltonian systems and non-hyperbolic dynamical systems [7, 8]. It is a remarkable problem that the theoretical meaning of the $1/f$ power spectrum has not been completely elucidated.

In ergodic theory the time average can be replaced by the ensemble average. However, in the intermittent phenomena, this replacement is not always guaranteed. Actually, it has been pointed out that the time average of some observation shows anomalous behaviour,

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that is, the time average does not converge to a constant value and becomes intrinsically random [9, 10]. Therefore it is important to analyze the behaviour of the time average in the intermittent phenomena from the viewpoint of ergodic theory. Infinite measure dynamical systems are examples of dynamical systems describing such intermittent phenomena. The recent study of infinite ergodic theory tells us that the time average of some observation functions converges in distribution [11–14]. For example, the scaled time average of the $L^1(m)$ function $g(x)$, $\sum_{k=0}^{n-1} g \circ T^k / a_n$, converges to the Mittag-Leffler distribution, where a_n is the proper sequence.

There are two purposes in this paper. One is to elucidate the non-stationary intermittent phenomena on the basis of the infinite ergodic theory. Another is to construct the ergodic measure in non-equilibrium state. For these purposes, we study the time average of the non-$L^1(m)$ function in infinite measure systems. In the previous works, it is already known that the distribution of the occupation time of some interval in infinite measure systems converges to the generalized arcsine distribution [15, 16]. In this paper we clear the class of the observation function whose time average converges to the generalized arcsine distribution using the typical example of a one-dimensional infinite measure dynamical system on [0, 1]. This study can be applied to the non-stationary time series, such as the intensity of the fluorescence in nanocrystals.

The paper is organized as follows. In Sect. 2 we introduce the one-dimensional map as a model of intermittent phenomena. In Sect. 3 we show the distribution for the time average of the $L^1_{loc,m}(0, 1)$ function with finite mean using Lamperti-Thaler limit theorem. Theorem 2, which is one of the main results in this paper, generalizes the observation function. In Sect. 4 our results are applied to the correlation functions. In Sect. 5 we numerically demonstrate the distribution for the time average of the $L^1_{loc,m}(0, 1)$ function with infinite mean. Section 6 is devoted to summary and an approach toward the ergodic problems of non-equilibrium statistical mechanics.

2 Intermittent Phenomena and Infinite Measure Dynamical Systems

Power law phenomena are clearly observed in on-off intermittent phenomena. When the mean of the duration time of on or off state is infinite, these intermittent phenomena can be described by infinite measure dynamical systems. Actually, dynamical systems have the invariant density $\rho(x) \sim x^{-1/(\beta-1)}$ when the probability density function (p.d.f.) $f(n)$ of the duration time n in the neighborhood of the indifferent fixed point ($x = 0$) is given by $f(n) \propto n^{-\beta}$. Therefore the invariant density $\rho(x)$ can not be normalized when the mean of the duration time is infinite, namely $\beta \le 2$.

In this paper we focus on the skew modified Bernoulli map defined by

$$x_{n+1} = T x_n = \begin{cases} x_n + (1-c)\left(\dfrac{x_n}{c}\right)^B & x_n \in I_0 = [0, c] \\[2ex] x_n - c\left(\dfrac{1-x_n}{1-c}\right)^B & x_n \in I_1 = (c, 1] \end{cases} \tag{1}$$

with $0 < c < 1$ as a typical phenomenological model of the intermittent phenomena. In the case of $c = 1/2$, the invariant density $\rho(x)$ is symmetric with respect to the axis $x = 1/2$ and can be written as

$$\rho(x) \sim x^{1-B} + (1-x)^{1-B} \tag{2}$$

for $x \sim 0$ and $x \sim 1$. On the other hand, in the case of $c \neq 1/2$, the invariant density is not symmetric. In both cases the invariant density cannot be normalized for $B \geq 2$, that is, the invariant measure becomes the infinite one [7, 17, 18].

It is important that renewal processes are constructed by the sequences of the skew modified Bernoulli map [19]. Actually, using the symbolic sequence $\sigma_n = \sigma(x_n)$, where $\sigma(x) = -1$ $(x \in I_0)$ and $\sigma(x) = 1$ $(x \in I_1)$, one can define the renewal time n when the value of σ_n changes, namely $\sigma_n \sigma_{n+1} = -1$. As the rth duration time of on state $(\sigma(x) = +1)$ or off state $(\sigma(x) = -1)$ $(r \geq 2)$ is an independently identically distributed random variable whose probability density function (p.d.f.) $f(n)$ is given by

$$f(n) \propto (n-1)^{-\beta} \quad (\beta = B/(B-1)) \tag{3}$$

for $n \gg 2$ [20], this process is regarded as a renewal process. In this model, the exponent of the p.d.f. for the on state is the same as that for the off state, but the scaling is not the same one.[1]

In the previous papers, it is pointed out that the p.d.f. of the first renewal time $\mathbf{X}_1$ depends on the initial ensemble of the modified Bernoulli map $(c = 1/2)$ [22, 23]. When the initial ensemble is the invariant density for the first passage map $T^{n(x)}(x)$ with respect to $E = [e_1, e_2]$, whose endpoints are the solutions of the equation $Tx = 1/2$ for $e_1 < e_2$ and $n(x) = 1 + \min\{n \geq 0 : T^n(x) \in E\}$, the p.d.f. of $\mathbf{X}_1$ is same as (3), namely the ordinary renewal process. However, when the initial ensemble is the invariant density for the map, the p.d.f. of the first renewal time $\mathbf{X}_1$ is given by $f_1(n) = (1 - F(n))/\mu$, which is completely different from (3), where $F(n)$ is the cumulative distribution function of $f(n)$ and μ is the mean value of $\mathbf{X}_r$ [19]. In [22, 23] we clearly demonstrate the dependence of the statistical laws, namely, the renewal function and the correlation function, on the initial ensemble.

3 Generalized Arcsine Law

Firstly, we analyze the behaviour of the time average of the following function:

$$I(x) = \begin{cases} a & (x \leq c) \\ b & (x > c), \end{cases} \tag{4}$$

where $a, b \in \mathbb{R} \setminus \{-\infty, \infty\}$.[2] It is noted that $I(x)$ is not the $L^1(m)$ function.

We review Lamperti-Thaler generalized arcsine law for the skew modified Bernoulli map [15, 25].

Theorem 1 (Lamperti-Thaler generalized arcsine law) *Let T be the skew modified Bernoulli map, then*

$$\lim_{n\to\infty} \Pr\left(\frac{1}{n}\sum_{k=0}^{n-1} 1_{[0,c]} \circ T^k \leq t\right) = G_{\alpha_1,\alpha_2}(t), \tag{5}$$

[1] When the exponent is different, statistical quantities are determined by the larger exponent [21]. That is, the p.d.f. of the on state is different from that of the off state. Therefore the resulting process is an alternating renewal process.

[2] The special case, namely $a = -1$, $b = 1$ and $c = 0.5$, is shown by using the renewal theory in [24].

where $\alpha_1 = \beta - 1$,

$$\alpha_2 = \frac{1+(B-1)c}{1+(B-1)(1-c)}\left(\frac{1-c}{c}\right)^{\frac{2}{B-1}} \tag{6}$$

and the p.d.f. $G'_{\alpha_1,\alpha_2}(t)$ *is given by*

$$G'_{\alpha_1,\alpha_2}(t) = \frac{\alpha_2 \sin\pi\alpha_1}{\pi} \frac{t^{\alpha_1-1}(1-t)^{\alpha_1-1}}{\alpha_2^2 t^{2\alpha_1} + 2\alpha_2 t^{\alpha_1}(1-t)^{\alpha_1}\cos\pi\alpha_1 + (1-t)^{2\alpha_1}}. \tag{7}$$

The distribution $G_{\alpha_1,\alpha_2}(t)$ is called the generalized arcsine distribution.

Using Theorem 1, one can know the distribution of the time average of $I(x)$ immediately.

Lemma 1 *The time average of* $I(x)$ *converges in distribution*:

$$\frac{1}{n}\sum_{k=0}^{n-1} I\circ T^k \to Y_{\alpha_1,\alpha_2,a,b} \quad \text{as } n\to\infty, \tag{8}$$

where the random variable $Y_{\alpha_1,\alpha_2,a,b}$ *has the p.d.f.* $G'_{\alpha_1,\alpha_2}(\frac{t-b}{a-b})$ *for* $a > b$ *and* $-G'_{\alpha_1,\alpha_2}(\frac{t-b}{a-b})$ *for* $a < b$.

Proof The time average of $I(x)$ can be rewritten as

$$\frac{1}{n}\sum_{k=0}^{n-1} I\circ T^k = \frac{aN_n + b(n-N_n)}{n}, \tag{9}$$

where N_n is the total occupation time in $[0, c]$. Using Theorem 1, we can write

$$\Pr\left\{\frac{1}{n}\sum_{k=0}^{n-1} I\circ T^k \le t\right\} = \Pr\left\{(a-b)\frac{N_n}{n} + b \le t\right\}$$
$$\to \begin{cases} G_{\alpha_1,\alpha_2}\left(\frac{t-b}{a-b}\right) & (a>b) \\ 1 - G_{\alpha_1,\alpha_2}\left(\frac{t-b}{a-b}\right) & (a<b) \end{cases} \quad \text{as } n\to\infty. \tag{10}$$

□

Definition 1 ($L^1_{loc,m}$ function with finite mean) If the following conditions

$$\lim_{\epsilon\to 0}\frac{\int_\epsilon^{1-\epsilon}|g|dm}{\int_\epsilon^{1-\epsilon}dm} < \infty \tag{11}$$

and for all $\epsilon > 0$

$$\int_\epsilon^{1-\epsilon}|g|dm < \infty \tag{12}$$

hold, then the function g is called the $L^1_{loc,m}(0,1)$ function with finite mean.

Let m be the invariant measure of the skew modified Bernoulli map, then examples of the $L^1_{loc,m}(0,1)$ function are $I(x)$ and $g(x)=x$.[3]

Remark 1 In the case of the skew modified Bernoulli map, the measure of the sets $[0,\epsilon]$ and $[1-\epsilon,1]$ are not finite. Therefore we exclude these sets in (11) and (12). When the measure of the other sets are not finite, Inequalities (11) and (12) must be changed to exclude these sets.

Theorem 2 *Let T and $g(x)$ be the skew modified Bernoulli map and the $L^1_{loc,m}(0,1)$ function with finite mean, respectively, and $g(0)=a$, $g(1)=b$. Further, there exist δ_1 and δ_2 such that $0<\delta_1,\delta_2<1$ and $g(x)$ is continuous in $[0,\delta_1]\cup[1-\delta_2,1]$. Then the time average of $g(x)$ converges in distribution to $Y_{\alpha_1,\alpha_2,a,b}$, where α_1 and α_2 are as in Theorem 1.*[4]

Proof It is shown that the time average of the $L^1_+(m)$ function[5] converges to zero [12], that is, for all $f\in L^1_+(m)$

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x)\to 0 \quad \text{as } n\to\infty \text{ for almost all } x. \tag{13}$$

By (13), for all $\epsilon>0$ there exists N_1 such that for $n>N_1$

$$\left|\frac{1}{n}\sum_{k=0}^{n-1} g_{\delta,a,b}(T^k x)-\frac{1}{n}\sum_{k=0}^{n-1} I(T^k x)\right|<\frac{1}{n}\sum_{k=0}^{n-1}|g_{\delta,a,b}(T^k x)-I(T^k x)|<\epsilon/2, \tag{14}$$

where

$$g_{\delta,a,b}(x)=\begin{cases} a & \text{for } x\in[0,\delta)\\ g(x) & \text{for } x\in[\delta,1-\delta]\\ b & \text{for } x\in(1-\delta,1]. \end{cases} \tag{15}$$

Then by the continuity of $g(x)$, for all $\epsilon>0$ there exists δ_* such that

$$\left|\frac{1}{n}\sum_{k=0}^{n-1} g(T^k x)-\frac{1}{n}\sum_{k=0}^{n-1} g_{\delta_*,a,b}(T^k x)\right|<\epsilon/2. \tag{16}$$

[3]Roughly speaking, we can say the $L^1_{loc,m}(0,1)$ function with finite mean is considered as the $L^\infty(0,1)$ function.

[4]The following proof is not changed even if the condition of $g(x)$ changes from the $L^1_{loc,m}(0,1)$ function to the $L^\infty(0,1)$ function.

[5]The $L^1_+(m)$ function is the $L^1(m)$ function whose value is positive.

Therefore for all $\epsilon > 0$ there exist δ_* and N such that for $n > N$

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} g(T^k x) - \frac{1}{n} \sum_{k=0}^{n-1} I(T^k x) \right| < \epsilon. \tag{17}$$

By Lemma 1, the time average of $g(x)$ converges in distribution to $Y_{\alpha_1,\alpha_2,a,b}$. □

Remark 2 The convergence process to the generalized arcsine distribution depends on δ_1, δ_2 and the derivatives of the function $g(x)$ near $x = 0$ and $x = 1$. Because of $g'(1) > g'(0)$, the p.d.f. of the numerical simulation in Fig. 1 has minor deviations from theoretical curve.

Figures 1, 2, 3, 4, 5, 6 and 7 demonstrate numerically that the probability density functions for the time average of $g(x) = x^2$ (Figs. 1, 2, 3, 4, 5 and 6) and $g(x) = 2^x$ (Fig. 7) obey the theoretical one (solid line) even when the time n is finite ($n = 10^8$).

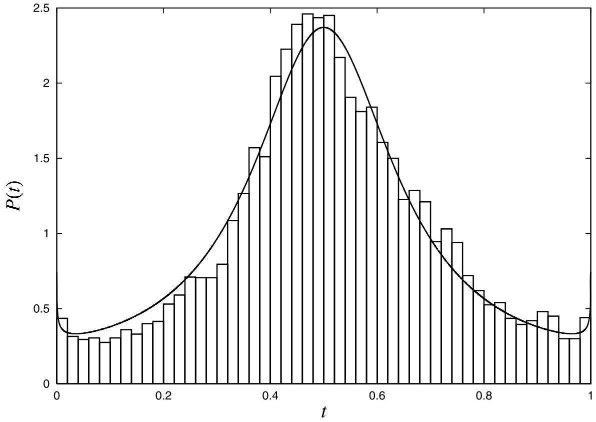


Fig. 1 The probability density function $P(t)$ for the time average of $g(x) = x^2$ ($B = 2.2, c = 0.5$)

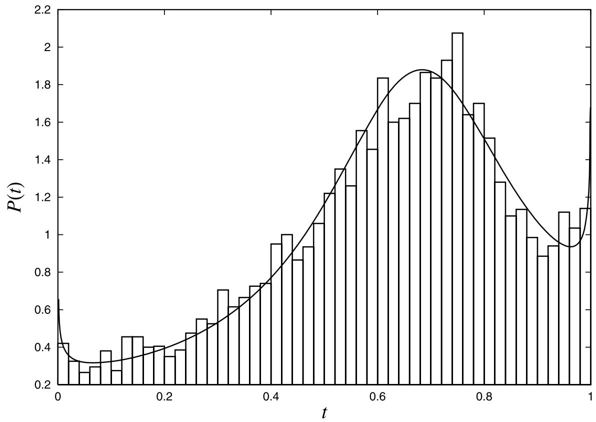


Fig. 2 The probability density function $P(t)$ for the time average of $g(x) = x^2$ ($B = 2.3, c = 0.4$)

Fig. 3 The probability density function $P(t)$ for the time average of $g(x) = x^2$ ($B = 2.5, c = 0.5$)

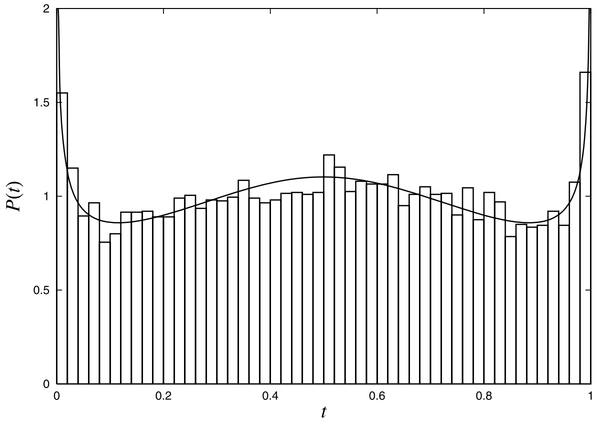


Fig. 4 The probability density function $P(t)$ for the time average of $g(x) = x^2$ ($B = 2.5, c = 0.1$)

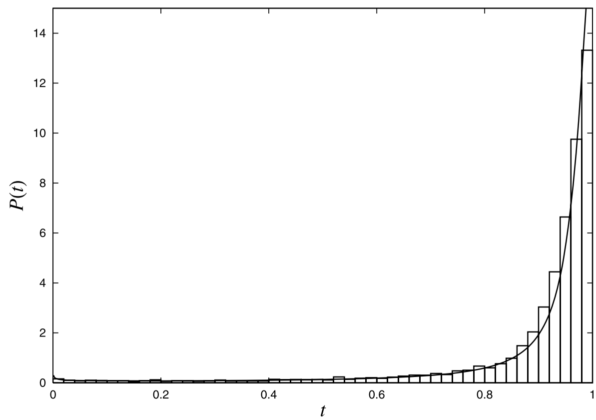
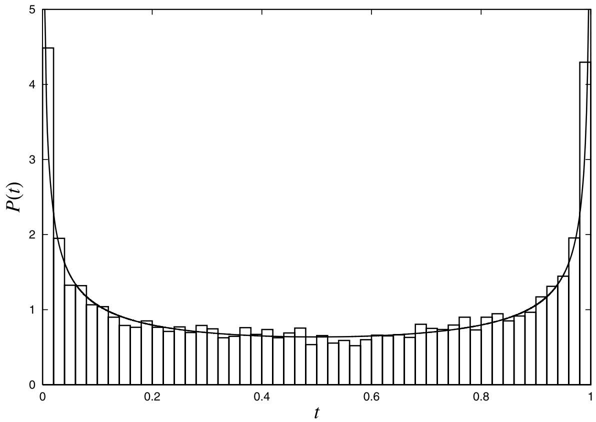


Fig. 5 The probability density function $P(t)$ for the time average of $g(x) = x^2$ ($B = 3.0, c = 0.5$)



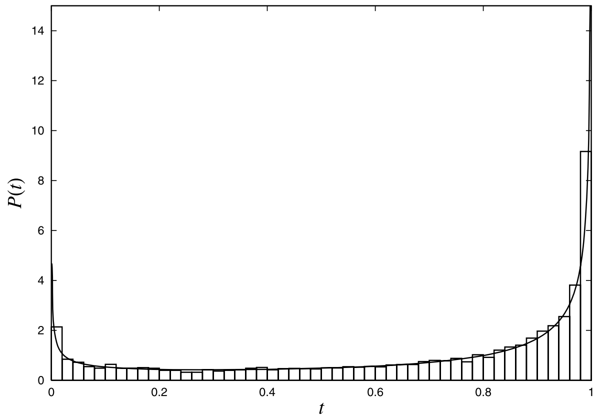


Fig. 6 The probability density function $P(t)$ for the time average of $g(x)=x^2$ $(B=3.0, c=0.2)$

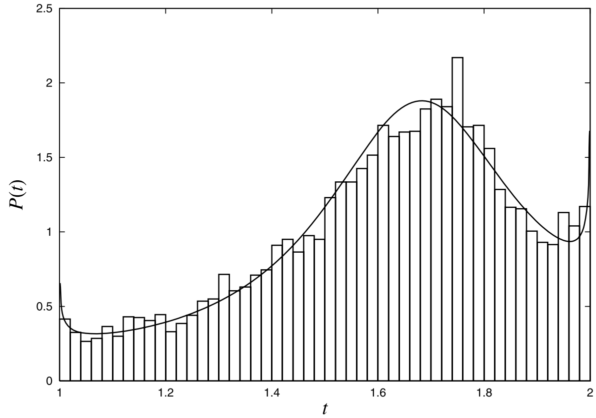


Fig. 7 The probability density function $P(t)$ for the time average of $g(x)=2^x$ $(B=2.3, c=0.4)$

4 Application to the Correlation Functions

We apply Theorem 2 to the correlation function, which is defined by the time average:

$$C(n)=\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}g(x_k)h(T^n x_k). \tag{18}$$

In the case of finite measure, ergodic theory states that the correlation function defined by the time average equals to the correlation function defined by the ensemble average, namely the average of $g(x)h(T^n x)$ with respect to the invariant measure. However, when the invariant measure is not finite, the time-averaged correlation function is not equal to the ensemble-averaged correlation function.[6] In this section we demonstrate that the correlation function is intrinsically random in the skew modified Bernoulli map.

[6] The dependence of the decay of the ensemble-averaged correlation function on the initial ensemble is discussed in [23].

Corollary 1 *For all n the correlation function of $\sigma(x)$ converges to* 1:

$$C(n) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \sigma(x_k)\sigma(x_{k+n}) = 1, \tag{19}$$

where $\sigma(x) = 1\ (x \in [0, c)),\ -1\ (x \in [c, 1])$.

Proof For all n the observation function $g_n(x)$ is defined as

$$g_n(x) = \sigma(x)\sigma(T^n x) = \begin{cases} +1 & x \in [0, a_n^+] \cup [1 - a_n^-, 1] \cup A_n \\ -1 & \text{otherwise}, \end{cases} \tag{20}$$

where $a_n^+ = a_{n+1}^+ + (1-c)(a_{n+1}^+/c)^B$ ($a_n^+ > 0$ and $a_0^+ = c$), $a_n^- = a_{n+1}^- - c(\frac{1-a_{n+1}^-}{1-c})^B$ ($a_n^- < 1$ and $a_0^- = c$) and A_n is the set which attains $\sigma(x)\sigma(T^n x) = 1$ and is subset of $[a_n^+, 1 - a_n^-]$:

$$A_n = \{x \in [a_n^+, 1 - a_n^-] : \sigma(x)\sigma(T^n x) = 1\}. \tag{21}$$

Then $g_n(0) = g_n(1) = 1$ and $g_n(x)$ is continuous in $[0, a_n^+] \cup [1 - a_n^-, 1]$ and for all $\epsilon > 0$

$$\int_{\epsilon}^{1-\epsilon} g_n(x) dm < \int_{\epsilon}^{1-\epsilon} dm < \infty. \tag{22}$$

By Theorem 2, $C(n)$ is convergence in distribution to $Y_{\alpha_1,\alpha_2,1,1}$ for all n. □

Corollary 2 *For all n the correlation function of x is convergence in distribution to $Y_{\alpha_1,\alpha_2,0,1}$*:

$$C(n) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} x_k x_{k+n} \to Y_{\alpha_1,\alpha_2,0,1}. \tag{23}$$

Proof For all n the observation function $g_n(x)$ is defined as

$$g_n(x) = x(T^n x), \tag{24}$$

and for all $\epsilon > 0$

$$\int_{\epsilon}^{1-\epsilon} g_n(x) dm < \int_{\epsilon}^{1-\epsilon} dm < \infty. \tag{25}$$

Then $g_n(0) = 0$ and $g_n(1) = 1$, and $g_n(x)$ is continuous in $[0, a_n^+] \cup [1 - a_n^-, 1]$. By Theorem 2, $C(n)$ is convergence in distribution to $Y_{\alpha_1,\alpha_2,0,1}$ for all n. □

Figures 8 and 9 show that the correlation function with fixed time difference n ($n = 10$ and 20) is intrinsically random and these distributions obey the generalized arcsine distribution.

Fig. 8 The probability density function $P(t)$ for the correlation function defined by the time average of $g_{10}(x) = xx_{10}$ ($B = 2.4$, $c = 0.3$ and $N = 10^7$)

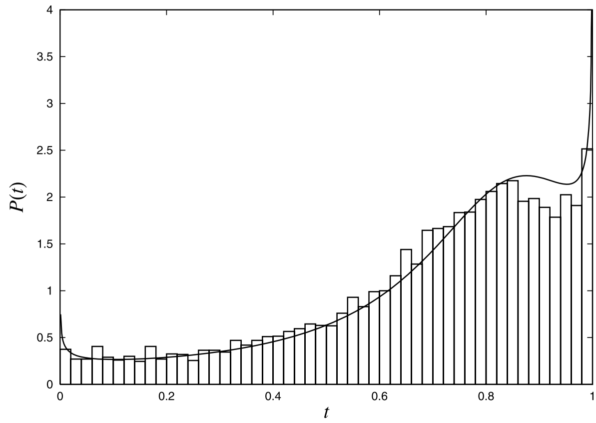
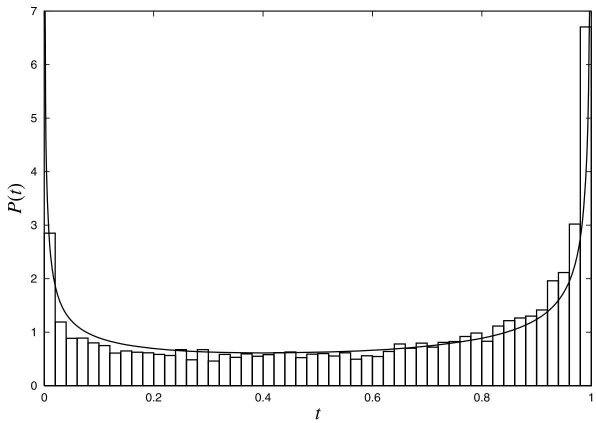


Fig. 9 The probability density function $P(t)$ for the correlation function defined by the time average of $g_{20}(x) = xx_{20}$ ($B = 3.0$, $c = 0.4$ and $N = 10^7$)



5 Stable Law in the Modified Bernoulli Map ($c = 1/2$)

Here we demonstrate the distribution for the time average of the $L^1_{loc,m}(0,1)$ function with infinite mean, which satisfies the condition (12) and

$$\lim_{\epsilon\to 0} \frac{\int_\epsilon^{1-\epsilon} |g| dm}{\int_\epsilon^{1-\epsilon} dm} = \infty. \tag{26}$$

In the case of the finite measure, the time average of the $L^1_{loc,m}(0,1)$ function with infinite mean converges to the stable distribution, as shown in Appendix A.

In what follows, we study numerically the distribution for the time average of the observation function

$$g(x) = \begin{cases} x^{-\alpha} & \left(x < \dfrac{1}{2}\right) \\ (1-x)^{-\alpha} & \left(x \geq \dfrac{1}{2}\right), \end{cases} \tag{27}$$

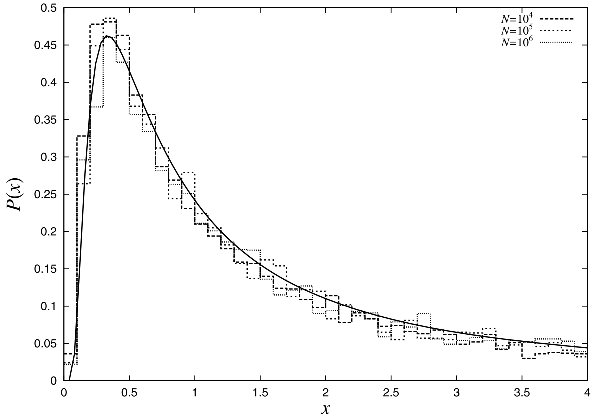


Fig. 10 The probability density function $P(t)$ for the scaled time average of $g(x)$ ($B = 3.0$ and $\alpha = 2.0$). The fitting curve is a stable distribution with $\gamma = 2.0$ ($P(x) = \frac{1}{\sqrt{2\pi x^3}} e^{-1/(2x)}$)

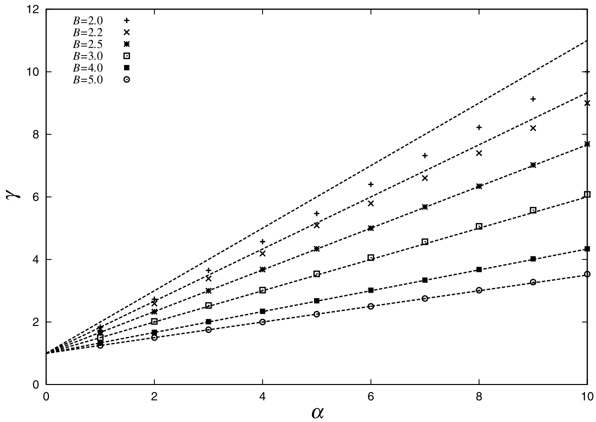


Fig. 11 Scaling exponent γ as a function of α. The *dashed lines* represent the relationship (28)

which is the $L^1_{loc,m}(0, 1)$ function with infinite mean for $\alpha > 0$, in the modified Bernoulli map ($c = 1/2$). As shown in Fig. 10, we find that the distribution for the scaled time average of $L^1_{loc,m}(0, 1)$ function with infinite mean also converges to the stable distribution with index $1/\gamma$. In numerical simulations we calculate the time average of $g(x)$ for three different length of the simulation time ($n = 10^4$, 10^5 and 10^6) and then determine the exponent γ so as to make the distributions of the scaled time average invariant. Figure 11 shows the linear relation between γ and α clearly. Moreover, as shown in Fig. 11, we find that the scaling exponent γ obeys the non-trivial relationship between exponents α, γ and B:

$$\gamma = \frac{\alpha}{B-1} + 1, \tag{28}$$

except for the case $B < 2.5$.[7] We summarize these results as the following conjecture.

[7] The reason that the relationship (28) does not hold for the case $B < 2.5$ seems to be that the observation time is not enough in numerical simulations. The theoretical argument is carried out in Appendix B.

Conjecture 1 *Let $g(x)$ be the $L^1_{loc,m}(0,1)$ function with infinite mean. Further,*

$$x^\alpha g(x) = O(1), \quad x \to 0 \tag{29}$$

$$(1-x)^\alpha g(x) = O(1), \quad x \to 1. \tag{30}$$

Then the scaled time average of $g(x)$ converges to the stable distribution $G_{1/\gamma}$:

$$\lim_{n\to\infty} \Pr\left(\frac{1}{b_n}\sum_{k=0}^{n-1} g\circ T^k \le t\right) = G_{1/\gamma}(t), \tag{31}$$

where $b_n \propto n^\gamma$ and $\gamma = \frac{\alpha}{B-1} + 1$.

Remark 3 In the case of $B = 2.0$ the exponent γ does not seem to hold the relationship (28) even when the observation time n is considerably large. The logarithmic correction of the scaling sequence b_n such as $n^\gamma/(\log n)^\alpha$ will be needed.

Remark 4 The distribution for the scaled time average of $g(x) = x^{-\alpha}$ also converges to the stable distribution when the invariant measure is finite. On the other hand, the distribution for the scaled time average of $g(x)$ in the case of the infinite measure is different from the stable one due to the generalized arcsine law for the occupation time of the interval $[1/2, 1]$:

$$\lim_{n\to\infty} \Pr\left(\frac{1}{b_n}\sum_{k=0}^{n-1} g\circ T^k \le t\right) = \frac{1}{2}\delta(t) + \frac{1}{2}G_{1/\gamma}(t). \tag{32}$$

6 Discussion

In this paper we present the distributional limit theorems for the time average of the $L^1_{loc,m}(0,1)$ function with finite mean and infinite mean using the map whose invariant measure is infinite. By applying the theorem to the correlation function, it is clearly shown that the correlation functions with fixed time difference n converge to the generalized arcsine distribution. G. Margolin and E. Barkai analyzed the distribution for the correlation function of a dichotomous random process and its convergence process [10]. Our results correspond to the generalized result of their work in the way that the observation function can be extended to the $L^1_{loc,m}(0,1)$ function with finite mean. That is, our results can be applied to not only the dichotomous process but also the temporal sequence itself.[8] However, the convergence process of the time average of the $L^1_{loc,m}(0,1)$ with finite mean is not studied. The analysis will be carried out using the assumption (38) and the property of the observation function near fixed points.

From the viewpoint of physical observation, the distributional limit theorems, namely, the Mittag-Leffler distribution and the generalized arcsine law and the stable law, suggest that one can characterize the behaviour of non-stationary phenomena through the distribution of the time average. It is important to know what class the observation function belongs to, that is, whether the observation function is the $L^1_{loc,m}(0,1)$ function with finite mean or

[8] In [10] the observation function is the characteristic function $I(x)$ with $a = 1$ and $b = 0$, which is the special case of our results.

Table 1 Universal distributions of the time average of the observation function $g(x)$

Invariant measure	$g(x)$	Distribution
Finite ($B < 2$)	$L^1(m)$	Delta
Finite ($B < 2$)	$L^1_{loc}(m)$ with infinite mean	Stable
Infinite ($B \geq 2$)	$L^1(m)$	Mittag-Leffler
Infinite ($B \geq 2$)	$L^1_{loc,m}$ with finite mean	Generalized arcsine
Infinite ($B \geq 2$)	$L^1_{loc,m}$ with infinite mean	Stable

not. Because the observation function in physical systems is not always the $L^1(m)$ function. Actually, we show that the correlation function is a typical example of the $L^1_{loc,m}(0,1)$ function with finite mean. This means that the correlation function, or the statistical quantities based on the time average in the intermittent phenomena, is intrinsically random. Universal distributions for the time average of various observation functions are shown in Table 1.

In the concept of the "ergodicity" proposed by L. Boltzmann, the ergodicity, i.e., the time average equals to the space average, guarantees the existence of the equilibrium state in dynamical systems on the assumption that macroscopic observables result from the time average of the microscopic observation function. In the non-equilibrium steady state, Sinai-Ruelle-Bowen measure is considered to describe the non-equilibrium steady state. However, there are no concepts of the "ergodicity" in the non-equilibrium state on the above assumption, that is, the measure of the non-equilibrium non-stationary state is not at all elucidated on the basis of the time average of the dynamical systems.[9] We hope that randomness of the time average in infinite measure systems will give us the motive argument toward the ergodic problem in the non-equilibrium state. There is a possibility that Mittag-Leffler distribution or the generalized arcsine distribution or the stable distribution could become one of the measures characterizing the non-equilibrium state. Actually, these distribution universally appear in diffusion and its generalizations. Moreover, the generalized arcsine law has drawn much attention in disordered systems [27, 28].

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Appendix A: The Distribution for the Time Average of the Non-$L^1(m)$ Function in the Case of the Finite Measure

In the case of the finite measure ($B < 2$) with $c = 1/2$, the invariant density can be written as

$$\rho(x) = \frac{2-B}{2}\{x^{1-B} + (1-x)^{1-B}\}. \tag{33}$$

Birkhoff's ergodic theorem [29] tells us that the p.d.f. of the sequence $\{Tx, T^2x, \ldots, T^nx\}$ obeys the invariant density as $n \to \infty$. Let $\mathbf{X}_n$ be random variables with p.d.f. (33) and

[9] From the aspect of the ensemble average, the approach to equilibrium of the observables is clearly shown using infinite measure systems in [22, 23, 26].

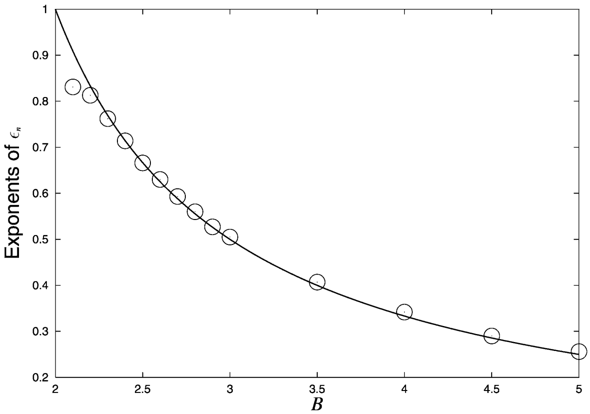

Fig. 12 The exponents of ϵ_n in (38) as a function of B. *Circles* are the results of numerical simulations, and the *solid line* represents $\frac{1}{B-1}$. In numerical simulations, we estimate the time n in (38) at $n = \min\{k \ge 1 : \int_0^{\epsilon_n} P^k 1 dx > 0.1\}$

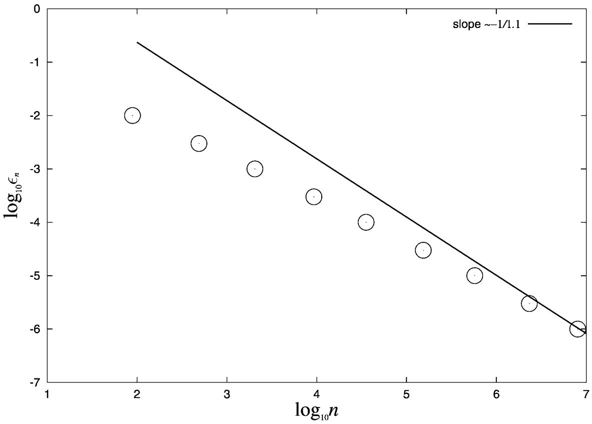

Fig. 13 The scaling law of ϵ_n ($B = 2.1$). The scaling law (38) is observed when ϵ_n is relatively small. The slope of the *solid line* is $-1/(B-1)$

$g(x) = x^{-\alpha}$ ($\alpha \ge 2 - B$). The distribution of $\mathbf{Y}_n = g(\mathbf{X}_n)$ is given by

$$\Pr(\mathbf{Y} < y) = \Pr(\mathbf{X} > y^{-\frac{1}{\alpha}}) = 1 - \frac{1}{2} y^{-\frac{2-B}{\alpha}} (1 - y^{-\frac{B-1}{\alpha}}). \tag{34}$$

Therefore

$$1 - \Pr(\mathbf{Y} < y) \sim \frac{1}{2} y^{-\frac{2-B}{\alpha}}, \quad y \to \infty. \tag{35}$$

The general central limit theorem [30] says that the random variable $(\mathbf{Y}_1 + \cdots + \mathbf{Y}_n)/b_n$ has the stable distribution, where $b_n = \Gamma(1 - (2-B)/\alpha) n^{\alpha/(2-B)}/2$:

$$\lim_{n\to\infty} \Pr\left(\frac{1}{b_n} \sum_{k=0}^{n-1} g \circ T^k < t \right) = G_{(2-B)/\alpha}(t), \tag{36}$$

where $G_{(2-B)/\alpha}(t)$ is the stable distribution with index $(2-B)/\alpha$.

Appendix B: The Scaling Exponent (28)

Let P be the Perron-Frobenius operator. We assume that for $u \in L^1(0,1)$ and $\int_0^1 u(x)dx = 1$

$$P^n u = \begin{cases} \frac{1}{2\epsilon_n} & (\epsilon_n \geq x) \\ 0 & (\epsilon_n < x < 1 - \epsilon_n) \\ \frac{1}{2\epsilon_n} & (x \geq 1 - \epsilon_n) \end{cases} \tag{37}$$

and

$$\epsilon_n \propto n^{-\frac{1}{B-1}}. \tag{38}$$

We have

$$\langle g \circ T^n \rangle = \int_0^1 g(x) P^n u(x) dx = O(\epsilon_n^{-\alpha}) = O(n^{\frac{\alpha}{B-1}}). \tag{39}$$

Therefore

$$\sum_{k=0}^{n-1} \langle g \circ T^k \rangle \propto n^{\frac{\alpha}{B-1}+1}. \tag{40}$$

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